Probability

I. Probability

A. Vocabulary

- **Probability** is the chance/likelihood of some event occurring.
  
  Ex) The probability of rolling a 1 for a six-faced die is \( \frac{1}{6} \). **It is read as “1 in 6” or “1 out of 6”.** In other words, you have a 1 in 6 chance (or a 1 out of 6 chance) of rolling a 1 when you roll the die.

- **Outcomes** are the possible results of an action. In other words, they are the possibilities.
  
  Ex) There are six outcomes for rolling a die: 1, 2, 3, 4, 5, and 6.

Check for Understanding

What is the meaning behind the following sentences? What is the difference between the phrases, “in theory” and “in actuality”?

“In theory, 30 students should be in class today. However, in actuality, only 28 students came to class today.”

II. Theoretical vs. Experimental Probability

- There are two types of probability:

  1) **Theoretical probability** is the chance of an event occurring in theory. In other words, it is what you expect to happen in a **perfect world**.

  2) **Experimental probability** is the probability of an outcome based on an experiment. In other words, it’s what happens in actuality or practice.

- Here are the two formulas for theoretical and experimental probability. Notice how they are basically the same thing.

  **Formula for Theoretical Probability**

  \[ P \text{ (event)} = \frac{\text{number of favorable outcomes}}{\text{total number of possible outcomes}} \]
Check for Understanding

1) What is the difference between theoretical probability and experimental probability?

Theoretical - what you think should happen.
Experimental - what actually happens.

2) If you run the coin flipping experiment 5,000 times, what can you expect the probability to be?

\[
\frac{2,500}{5,000} = \frac{1}{2}
\]

Directions: Find the probability for one roll of a die. Write the probability as a fraction.

Ex) \(P(3) = \frac{1}{6}\)  \(\frac{2,500}{5,000} = \frac{1}{2}\)  \(\frac{1}{2} \text{ (1 out of 2)}\)

Ex) \(P(3 \text{ or } 4) = \frac{2}{6} = \frac{1}{3}\)

Ex) \(P(\text{even}) = \frac{3}{6} = \frac{1}{2}\)

Ex) \(P(4) = \frac{1}{6}\)

Ex) \(P(1, 2, \text{ or } 3) = \frac{3}{6} = \frac{1}{2}\)

Ex) \(P(\text{not } 3 \text{ or } 4) = \frac{4}{6} = \frac{2}{3}\)

Directions: Find the probability for selecting a letter at random from the word ARKANSAS. Write the probability as a fraction.

Ex) \(P(K) = \frac{1}{8}\)

Ex) \(P(N) = \frac{1}{8}\)

Ex) \(P(\text{vowel}) = \frac{3}{8}\)

Ex) \(P(K \text{ or } N) = \frac{2}{8} = \frac{1}{4}\)

Ex) \(P(A \text{ or } S) = \frac{5}{8}\)

Ex) \(P(N, R, \text{ or } S) = \frac{4}{8} = \frac{1}{2}\)

A. Experimental Probability

- Please remember that there are two types of probability:

  1) **Theoretical Probability** is the chance of an event occurring in theory. In other words, it is what you expect to happen or what should happen in a perfect world.

  2) **Experimental Probability** is the probability of an outcome based on an experiment. In other words, it’s what actually happens in the real world.
Check for Understanding

*What is the difference between theoretical probability and experimental probability?

Examples

Ex) Suppose you toss a coin 60 times and get tails 25 times. What is the experimental probability of getting tails and the experimental probability of getting heads?

\[ P(\text{tails}) = \frac{25}{60} = \frac{5}{12} \]

\[ P(\text{heads}) = \frac{35}{60} = \frac{7}{12} \]

Ex) An employee at Toys ‘R’ Us checked 400 toy cars. He found 12 defective cars. What is the experimental probability a toy car is defective?

\[ P(\text{defective car}) = \frac{12}{400} = \frac{3}{100} \]

Ex) Suppose you go to the swap meet on Sunday. You ask 20 vendors if they make their products in Hawai’i. Only 6 vendors say that they make their products in Hawai’i. What is the experimental probability of buying from a vendor who makes his or her products in Hawai’i?

\[ P(\text{products made in Hawai’i}) = \frac{6}{20} = \frac{3}{10} \]

Ex) Chris shoots 50 free throws. If he makes 42 free throws, what is the experimental probability that he misses the basket?

\[ P(\text{misses the basket}) = \frac{8}{50} = \frac{4}{25} \]

B. Theoretical Probability

Examples

Ex) In a standard deck of 52 cards, there are 4 kings, 4 queens, and 4 jacks. What is the theoretical probability of randomly selecting one of these face cards from the deck?

A) \( \frac{1}{4} \)  
B) \( \frac{1}{2} \)  
C) \( \frac{3}{13} \)  
D) \( \frac{4}{13} \)

Ex) Sachiel has 5 orange and 10 green marbles in a bag. What is the theoretical probability of choosing an orange marble from the bag?

\[ \frac{5}{15} = \frac{1}{3} \]
Ex) Alina has 4 nickels, 2 dimes, and 6 quarters in her pocket. What is the theoretical probability that she will randomly select a quarter from her pocket? \[ \frac{6}{12} = \frac{1}{2} \]

A) \( \frac{1}{2} \)  
B) \( \frac{1}{3} \)  
C) \( \frac{1}{4} \)  
D) \( \frac{1}{6} \)

Ex) There are six forks, four spoons, and eight knives in a drawer. What is the theoretical probability that someone reaching into the drawer will randomly select a fork? \[ \frac{6}{18} = \frac{1}{3} \]

C. Theoretical Probability vs. Experimental Probability

Ex) Kat conducted a probability experiment by flipping a coin 100 times. She recorded her results as shown below:

- 30 heads
- 70 tails

Kat claimed that her results did not match the theoretical probability of flipping a coin. What should her results have been in order for the results to match the theoretical probability of flipping a coin?

A) 20 heads and 80 tails  
B) 40 heads and 60 tails  
C) 80 heads and 20 tails  
D) 50 heads and 50 tails

Ex) A weather reporter stated that the probability of rain last week as \( \frac{4}{7} \) days. It rained on Monday, Tuesday, Wednesday, Friday, and Saturday last week. How did the reporter’s stated probability for rain last week compare to the actual results?

A) The probability of the rain matched the actual results.  
B) The probability of rain was less than the actual results.  
C) The probability of rain was greater than the actual results.  
D) The probability of rain would have matched the actual results if it had rained on Wednesday.

Ex) Mr. Hayes tossed a coin 12 times to determine whether or not it would land on hands or tails. His results are below. Find the experimental probability of getting tails. Write your answer as a fraction, decimal, and a percent.

Fraction: \( \frac{5}{12} \)  
Decimal: \( 0.4167 \)  
Percent: \( 41.67\% \)

a) What is the theoretical probability that Mr. Hayes gets tails?

\[ \frac{6}{12} \]

b) Referring to this problem, which statement is true?

1) The theoretical probability is greater than the experimental probability.  
2) The experimental probability is greater than the theoretical probability.
Ex) Mr. Stout tossed a coin 10 times to determine whether or not it would land on heads or tails. His results are below. Find the experimental probability of getting heads. Write your answer as a fraction, decimal, and a percent.

T, T, H, T, T, H, H, H, T

Fraction: \( \frac{4}{10} \)  
Decimal: 0.4  
Percent: 40%

a) What is the theoretical probability that Mr. Stout gets heads?

\( \frac{5}{10} \)

b) Referring to this problem, which statement is true?

1) The theoretical probability is greater than the experimental probability.

2) The experimental probability is greater than the theoretical probability.

D. Applying Probability to Larger Situations & Settings

- For some problems, you are expected find the probability of something occurring and then apply it to a whole group of people or a situation in which many trials are run.
  - To solve these problems, you are going to have to take the probability you found and multiply it by the whole group of people or the number of trials that are run. It should look something like this:

  \[
  \text{Probability} \times \text{whole group of people}
  \]
  
  OR

  \[
  \text{probability} \times \text{the number of trials that are run}
  \]

Examples

Ex) In preparation for her ice cream party, Emily surveyed 35 students at Pioneer Middle School about their favorite ice cream flavor. She found out that 7 of them like vanilla. If 210 students are expected to attend the ice cream party, about how many will prefer vanilla?

\[
\frac{7}{35} = \left(\frac{1}{5}\right) \quad \rightarrow \quad \frac{1}{5} \times 210 = \frac{210}{5} = 42 \text{ students}
\]

Ex) Iyonna was chosen to dj the school dance. To prepare for the dance, Iyonna asked 40 students at Pionner Middle School what songs they liked. She found out that 8 of the students liked Black and Yellow. If 300 students are expected to attend the dance, about how many will like Black and Yellow?

\[
\frac{8}{40} = \left(\frac{1}{5}\right) \quad \rightarrow \quad \frac{1}{5} \times 300 = \frac{300}{5} = 60 \text{ students}
\]

Ex) A student is taking a multiple-choice test that has 80 questions. Each question has 5 answer choices. If the student guesses randomly on every question, how many questions should the student expect to answer correctly?

A) 12  B) 16  C) 32  D) 40

\[
\frac{1}{5} \times 80 = \frac{80}{5} = 16
\]
Ex) Alyssa is playing a game in which she rolls a number cube with sides labeled 1 through 6. She rolls the cube 24 times during the game. Based on the theoretical probability, how many times should she expect to roll a number less than 4?

\[ \frac{3}{6} \times \frac{24}{1} = \frac{72}{6} = 12 \]

A) 12  B) 14  C) 16  D) 24

Ex) A fair number cube with faces numbered 1 through 6 was rolled 20 times. The cube landed with the number 4 up 6 times. What is the difference between the experimental probability and the theoretical probability of the number 4 landing face up?

Theoretical: \( P(4) \) \( \frac{1}{6} \) \( \rightarrow \frac{1}{6} \times 20 = \frac{20}{6} = \text{about 3} \)

Experimental: \( P(4) = 6 \) \( \text{The experimental probability is about 3 more than theoretical.} \)

Ex) A fair number cube with faces numbered 1 through 6 was rolled 50 times. The cube landed with the number 2 up 10 times. What is the difference between the experimental probability and the theoretical probability of the number 2 landing face up?

Experimental: 10 times

Theoretical: \( \frac{1}{6} \times 50 = \frac{50}{6} = \text{about 8 times} \) \( \text{2 more than the theoretical.} \)

III. Independent vs. Dependent Events

Check for Understanding

1) What does “independent” mean?

Is not affected / does not affect other things

2) What does “dependent” mean?

Can be affected by others

A. Vocabulary

- **Independent events** are events for which the occurrence of one event does not affect the probability of the occurrence of the other.

**Probability of Independent Events**

\[ P(A, \text{ then } B) = P(A) \cdot P(B) \]
- **Dependent Events** are events for which the occurrence of one event affects the probability of the occurrence of the other.

### Probability of Dependent Events

\[
P(A, \text{then} \ B) = P(A) \cdot P(B \text{ after} \ A)
\]

For two dependent events A and B, the probability of both events occurring is the

*Notice how you are doing the same thing for both independent and dependent events. You are **multiplying** the first probability by the second probability.*

### B. Figuring out the Difference between Independent and Dependent Events

- To help you figure out whether you have an independent or dependent event, you should ALWAYS ask yourself the following question:
  
  
  ***Is the probability of the second event directly affected by the probability of the first event?***

- If the answer is no, then it is an **independent event**.
  
  - Here are some key words/ situations that will tell you that you have an independent event:
    
    1) Rolling a number cube(s) or flipping a coin(s)
    
    2) Taking an (item) out and replacing the (item)
    
    3) Taking an (item) out and putting the (item) back

- If the answer is yes, then it is a **dependent event**.
  
  - Here are some key words/ situation that will tell you that you have a dependent event:
    
    1) Taking an (item) out and not putting it back

    a. without replacing the (item)…

    b. without putting the (item) back…..

**Examples**

Directions: Please say whether the event is independent and dependent and explain why.

Ex) You roll a number cube. You roll it again.

\[\text{Independent because the outcome of the 2nd roll is not affected by the 1st roll.}\]
Ex) You select a card from a deck. Without putting the card back, you select a second card. 

Dependent because the 2nd outcome is affected by the first draw. (not returning the card)

Ex) You flip a coin two times. On your first flip, it lands on heads. What is the probability that the coin will land on heads on your second flip?

Independent

Ex) You have 10 marbles in a bag, of which 6 are red and 4 are blue. You pull a red marble randomly out of the bag. Without replacing the marble, you pull another marble out of the bag. What is the probability that the second marble will also be red?

Dependent

Ex) You pick a marble from a bag containing 2 blue marbles, 5 red marbles, and 3 purple marbles. You replace the marble and select a second marble.

Independent

Ex) You select a card from a deck. You put the card back in the deck and then select a second card.

Independent

Check for Understanding

1) Explain why this is a dependent event: You select a card randomly from a deck of 52 cards, and without putting the card back, you select another card from the deck.

This is a dependent event because when you don't replace the card on the first draw, it affects the outcome of the second draw.

2) Explain why this is an independent event: You pick a marble from a bag containing 20 marbles. You replace the marble and select a second marble.

This is independent because when you replace the marble from the first pull, it does not affect the outcome of the second pull.

C. Solving Independent and Dependent Events

• Before you start any of these problems, you need to figure out whether or not you have an independent or dependent event.

Examples

Directions: You roll a number cube (with sides labeled 1 through 6) twice. What is the probability that you roll each pair of numbers?

*Before you answer the questions below, is this an independent or dependent event?

Independent

Ex) P(6, then 5) 
\[ \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} \]

Ex) P(6, then 2) 
\[ \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} \]

Ex) P(1, then 2 or 5) 
\[ \frac{1}{6} \cdot \frac{2}{6} = \frac{2}{36} = \frac{1}{18} \]
Ex) $P(3, \text{ then a number less than 4})$

$\frac{10}{16} \cdot \frac{3}{10} = \frac{30}{160} = \frac{3}{16}\left(\frac{\text{red}}{\text{P}}\right)$

Ex) $P(\text{even, then 1 or 6})$

$\frac{3}{16} \cdot \frac{2}{10} = \frac{6}{160} = \frac{1}{25}$

Ex) $P(\text{even, odd})$

$\frac{3}{16} \cdot \frac{3}{10} = \frac{9}{160} = \frac{1}{4}$

Directions: You select a card at random from the cards below. Without replacing the card, you select a second card. Find the probability of selecting each set of letters.

\[
\begin{array}{cccccccc}
P & R & E & A & L & G & E & B & R & A
\end{array}
\]

*Before you answer the questions below, is this an independent or dependent event? 

**Dependent**

Ex) $P(P, \text{ then } G)$

$\frac{1}{10} \cdot \frac{1}{9} = \frac{1}{90}$

Ex) $P(B, \text{ then } A)$

$\frac{1}{10} \cdot \frac{2}{9} = \frac{2}{90} = \frac{1}{45}$

Ex) $P(G, \text{ then } A \text{ or } R)$

$\frac{1}{10} \cdot \frac{4}{9} = \frac{4}{90} = \frac{2}{45}$

Directions: You pick a marble from a bag containing 1 green marble, 4 red marbles, 2 yellow marbles, and 3 black marbles. You replace the first marble then select a second one. Find each probability.

*Before you answer the questions below, is this an independent or dependent event? 

**Independent**

Ex) $P(\text{red, then yellow})$

$\frac{4}{10} \cdot \frac{2}{10} = \frac{8}{100} = \frac{2}{25}$

Ex) $P(\text{red, then black})$

$\frac{4}{10} \cdot \frac{3}{10} = \frac{12}{100} = \frac{3}{25}$

Directions: Daniel has 5 blue socks, 4 black socks, and 3 white socks. He selects one sock at random. Without replacing the sock, he selects a second sock at random. Find each probability.

*Before you answer the questions below, is this an independent or dependent event? 

**Dependent**

Ex) $P(\text{black, then white})$

$\frac{4}{12} \cdot \frac{3}{11} = \frac{12}{132} = \frac{1}{11}$

Ex) $P(\text{white, then blue})$

$\frac{3}{12} \cdot \frac{5}{11} = \frac{15}{132} = \frac{5}{44}$

Ex) $P(\text{black, then black})$

$\frac{4}{12} \cdot \frac{3}{11} = \frac{12}{132} = \frac{1}{11}$

Ex) $P(\text{blue, then blue})$

$\frac{5}{12} \cdot \frac{4}{11} = \frac{20}{132} = \frac{5}{33}$

Ex) When two coins are tossed at the same time, what is the theoretical probability of an outcome of one head and one head?

A) $\frac{1}{4}$  B) $\frac{1}{3}$  C) $\frac{1}{2}$  D) $\frac{3}{4}$

\[\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}\]
Ex) When three coins are tossed at the same time, what is the theoretical probability of an outcome of one head, one tail, and one head?

A) $\frac{1}{4}$  B) $\frac{1}{3}$  C) $\frac{1}{2}$  D) $\frac{1}{8}$

\[
P(H) \text{ and } P(T) \text{ and } P(H) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}
\]

Ex) When two number cubes are rolled at the same time, what is the theoretical probability of rolling a 1 and 5?

A) $\frac{1}{2}$  B) $\frac{1}{36}$  C) $\frac{1}{8}$  D) $\frac{5}{36}$

\[
P(1) \text{ and } P(5) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}
\]

IV. Spinners

- A spinner is a circle that is divided up into different parts according to category. The categories can be numbers, letters, colors, money, etc. Spinners are commonly used in games involving probability.

  - A famous example of a spinner is the game show, *Wheel of Fortune!*

A. Problems involving Spinners

Ex) Mrs. Watkins is using the spinner below to play a game.

a) What is the theoretical probability of the spinner landing on the letter $B$?

\[
\frac{3}{8}
\]

b) If the pointer is spun a total of 80 times, how many times can she expect the pointer to land on the letter $B$?

\[
\frac{3}{8} \cdot 80 = \frac{240}{8} = \boxed{30 \text{ times}}
\]
Ex) Oneka is using the spinner below to play a game.

[Diagram of a spinner with numbers 1, 2, 3, 4, 5, and 6]

\[ \frac{2}{8} = \frac{1}{4} \]

a) What is the theoretical probability of the spinner landing on 1? \[ \frac{1}{4} \]

b) If the pointer is spun a total of 60 times, how many times can she expect the pointer to land on 1?

\[ \frac{1}{4} \times 60 = 15 \text{ times} \]

B. Problems involving Two Spinners

Ex) Daniella is using the two spinners below to play a game.

[Diagram of two spinners with numbers 1, 2, 3, 4, 5, 6, 9, and 12]

If Daniella spins both spinners at the same time, what is the probability that she spins a 3 for the first spinner and a 12 for the second spinner?

\[ P(3, \text{then } 12) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12} \]

Ex) Brian has the spinner shown below and a coin.

[Diagram of a spinner with numbers 1, 2, 3, 4]

For an experiment, he spins the spinner and then tosses the coin. He does the experiment 64 times. How many times should Brian expect to spin a 4 and then have the coin land on heads?

\[ \frac{1}{4} \times \frac{1}{2} = \frac{1}{8} \times 64 = 8 \text{ times} \]

Ex) Let’s say you toss a coin and roll a number cube 144 times. What is the expected result of getting tails and a number 4?

\[ P(T) \text{ and } P(4) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12} \times 144 = 12 \text{ times} \]
V. Sample Size

- A **sample** is a part of a group you use to make estimates about that group. The **larger** your **sample**, the more reliable your estimates will be.

- A **random sample** is when you select members of the group at random. It is likely to be **representative** of the whole group.

Here are two examples of taking a sample:

- Mrs. Watkins decides to randomly survey 10 out of the 90 students she teaches. The 10 students are considered the sample and their answers can be used to represent the entire 90 students she teaches.

- Mrs. Watkins is rolling a number cube 20 times to see how many times it will land on 6. The number of trials (20 times) represents the sample size. The more trials she runs (the larger the sample size), the more accurate her results will be.

A. Theoretical vs. Experimental Probability

- The more trials of an experiment you run, the closer your **experimental** probability will get to the **theoretical** probability.

- Similarly, the larger the **sample size** of an experiment, the more accurate the results will be. This means that the experimental probability will get closer to the theoretical probability (to be more accurate).

Examples

Ex) The expected probability of rolling an even number in 1 roll of a fair cube with faces numbered 1 through 6 is \( \frac{1}{2} \). When the cube was rolled 20 times, an even number came up 15 times, or \( \frac{3}{4} \) of the time. When the same cube was rolled 100 times, an even number came up 51 times, or almost \( \frac{1}{2} \) the time. Why are the actual results closer to the expected probability of \( \frac{1}{2} \) when rolling the cube 100 times?

A) A larger sample size was used.  
B) The 100 tosses were controlled better.  
C) The thrower disregarded the odd rolls.  
D) The cube was rolled an even number of times.

Ex) Twenty students choose a piece of fruit from a list of 4 fruits: apple, banana, grape, and pear. The theoretical probability that a student will choose a banana is 0.25. Only 1 student chooses a banana. How can the experimental probability get closer to the theoretical probability?

A) Use a larger sample size of students.  
B) Use a smaller sample size of students.  
C) Provide more choices of fruit.  
D) Only give two choices of fruit.
Ex) Fifty students choose a piece of fruit from a list of 4 fruits: apple, banana, grape, and pear. The theoretical probability that a student will choose an apple is 0.25. Only 1 student chooses an apple. How can the experimental probability get closer to the theoretical probability?

A) Use a larger sample size of students.  
B) Use a smaller sample size of students.  
C) Provide more choices of fruit.  
D) Only give two choices of fruit.

Ex) Mr. Nystrom is conducting an experiment which involves rolling a pair of number cubes numbered 1 through 6. Rolling the sum of 12 is predicted to occur only 1 out of every 36 times. If Mr. Nystrom has already rolled four 12s out of 10 rolls, how might he get his results closer to the predicted results?

A) Roll the number cubes 100 or more times.  
B) Roll the number cubes 10 more times.  
C) Shake the number cubes harder.  
D) Ignore the next 12s that occur.

Ex) Jayden tossed a fair number cube with faces numbered 1 through 6 a total of 30 times. The number 5 came up 8 times. How do Jayden’s results compare to the expected probability of getting the number 5 on 30 tosses?

A) The outcome of 5 occurred more often than expected.  
B) The outcome of 5 occurred less often than expected.  
C) The outcome of 5 occurred 7 more times than expected.  
D) The outcome of 5 occurred exactly the same number of times as expected.

Ex) Caleb tossed a fair number cube with faces numbered 1 through 6 a total of 36 times. The number 6 came up 10 times. How do Caleb’s results compare to the expected probability of getting the number 6 on 36 tosses?

A) The outcome of 6 occurred more often than expected.  
B) The outcome of 6 occurred less often than expected.  
C) The outcome of 6 occurred 7 more times than expected.  
D) The outcome of 6 occurred exactly the same number of times as expected.

Ex) The theoretical probability of tossing two heads when tossing a pair of coins is 0.25. When the pair of coins was tossed 20 times, two heads came up only 2 times. Which procedure would result in an experimental probability that is closer to the theoretical probability?

A) Toss the coins more than 20 times.  
B) Toss three coins instead of two coins.  
C) Toss the coins fewer than 20 times.  
D) Count only those tosses that result in two heads or two tails.

Ex) The theoretical probability of tossing two heads when tossing a pair of coins is 0.25. When the pair of coins was tossed 40 times, two heads came up only 5 times. Which procedure would result in an experimental probability that is closer to the theoretical probability?

A) Toss the coins more than 40 times.  
B) Toss three coins instead of two coins.  
C) Toss the coins fewer than 40 times.  
D) Count only those tosses that result in two heads or two tails.